DTIC FILE COPY

(2)

SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION

1b. RESTRICTIVE MARKINGS

2a. SECURITY CLASSIFICATION AUTHORITY

2b. DECLASSIFICATION / DOWNGRADING SCHEDULE

AD-A222 137

3. DISTRIBUTION / AVAILABILITY OF REPORT
Approved for public release; distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S)

5. MONITORING ORGANIZATION REPORT NUMBER(S)
AFOSR-TR- 90-0634

6a. NAME OF PERFORMING ORGANIZATION
Trustees of Boston Univ.

6b. OFFICE SYMBOL (If applicable)

7a. NAME OF MONITORING ORGANIZATION
AFOSR

6c. ADDRESS (City, State, and ZIP Code)
881 Commonwealth Ave.
Boston, MA 02215

7b. ADDRESS (City, State, and ZIP Code)
Bolling Air Force Base
DC 20332-6448

8a. NAME OF FUNDING / SPONSORING ORGANIZATION
AFOSR

8b. OFFICE SYMBOL (If applicable)
NM

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
AFOSR-89-0115

8c. ADDRESS (City, State, and ZIP Code)
AFOSR, Building 410
Bolling Air Force Base, DC 20332-6448

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
61102F	2304	A6	

11. TITLE (Include Security Classification)
Statistical Methodologies for non-Gaussian Noisy Environments

12. PERSONAL AUTHOR(S)
Murad S. Taqqu

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
FINAL	FROM 11/01/88 TO 4/30/90	May 9, 1990	10

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Long-range dependence, high variability, stable noise, non-linear filtering, non-Gaussian noise

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

SEE REVERSE

DTIC ELECTE MAY 21 1990 S B D

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS

21. ABSTRACT SECURITY CLASSIFICATION
Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL
Dr. Ivan H. Sloggen

22b. TELEPHONE (Include Area Code)
(202) 767-4940

22c. OFFICE SYMBOL
NM

DD Form 1473, JUN 86 Previous editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

AFOSR-89-0115
Final Report Documentation Page
Block 19

ABSTRACT

The research focused on long-range dependence, high variability and non-linear filtering in time series. Long-range dependence occurs when the low frequencies have a fundamental impact on the dependence structure of the data. There is high variability when the data is non-Gaussian and takes large values with high probability, for example when it is Pareto or has a stable distribution. The data can also be the output of a non-linear filter and hence possess non-linear characteristics.

We studied the behavior of symmetric statistics, e.g. U and Von Mises statistics, and the asymptotic behavior of quadratic forms, when the data has finite variance with long-range dependence. The results are non-standard. We also focused on situations where the data has a stable distribution and hence exhibits high-variability. We studied joint moments and conditional moments, and linear regression problems. Linear Fractional Lévy Motions are important models when the data exhibit both high variability and long-range dependence. We found that there are many such models and we investigated their asymptotic dependence structure. We studied the tail behavior of probability distributions of multiple integrals with respect to stable noise and the sample paths behavior of related stochastic processes. We also developed a technique for approximating both the information structure and the paths behavior of stochastic processes.

FINAL REPORT

TECHNICAL DESCRIPTION OF RESULTS

STATISTICAL METHODOLOGIES FOR NON-GAUSSIAN NOISY ENVIRONMENTS

Murad S.Taqqu

Principal investigator

Air Force Office of Sponsored Research AFOSR-89-0115

This is the final Report on the Air Force Office of Sponsored Research grant AFSOR-89-0115 at Boston University. Results of the completed work are summarized here. For ease of exposition the subsections below may include the description of more than one article.

Symmetric statistics of long-range dependent observation

Dehling and Taqqu (1988b) study the behavior of classical bivariate statistics when the observations have long-range dependence. They suppose that these observations are the output of a non-linear filter with Gaussian input.

Applications include the sample variance, the chi-squared goodness of fit test and the Cramer-von Mises ω^2 criterion. The results obtained are non-standard, illustrating the fact that classical statistics can have unusual properties when there is long-range dependence.

Continuous functions whose level sets are orthogonal to all polynomials of a given degree

The following result, obtained by Dehling and Taqqu (1988a) is used to settle an open problem on empirical processes of long-range dependent sequences. The authors provide an explicit construction of continuous functions whose level sets are orthogonal to all polynomials of a given degree.

Power counting theorems

For what values of $\alpha_0, \alpha_1, \alpha_2$ and α_3 does the integral

$$\int_{\mathbf{R}^3} \left| \frac{e^{i(x+y+z)-1}}{(x+y+z)} \right|^{\alpha_0} |x+1|^{\alpha_1} |y-2|^{\alpha_2} |z|^{\alpha_3} dx dy dz$$

converge? The power counting theorem obtained by Terrin and Taqqu (1989a) gives the answer. It provides convergence conditions for a class of integrals on $\mathbf{R}^n$ whose integrands are products of functions bounded near zero by powers of linearly dependent affine functionals and near infinity by different powers of those functionals. The theorem is useful in studying asymptotic distributions of statistics of time series with long range dependence. It is also related to Dyson's power counting theorem in mathematical physics.

A Noncentral limit theorem for quadratic forms of Gaussian stationary sequences

Let $X_0, X_1, X_2, \ldots$ be a mean zero stationary Gaussian sequence with spectral density $f(x) = |x|^{-\alpha} L_1(x)$, $\alpha < 1$ and let $\{a_i\}_{i=-\infty}^{\infty}$ be the Fourier coefficients of a function $g(x) = |x|^{-\beta} L_2(x)$, $\beta < 1$. Here L_1 and L_2 are slowly varying functions at 0, bounded on bounded intervals. The condition $\alpha < 1$ and $\beta < 1$ ensures that the Fourier coefficients of f and g are well defined. (The Fourier coefficients of f are the covariances $r(k) = EX_i X_{i+k}$, $k = -\infty, \ldots, +\infty$). When $\alpha > 0$, the stationary sequence exhibits "long-range dependence."

Terrin and Taqqu (1989b) investigate the limit of the normalized quadratic form

$$Z_N(t) = \frac{1}{d_N} \left\{ \sum_{j=0}^{[Nt]-1} \sum_{k=0}^{[Nt]-1} a_{j-k} X_j X_k - E \sum_{j=0}^{[Nt]-1} \sum_{k=0}^{[Nt]-1} a_{j-k} X_j X_k \right\} \qquad (1)$$

for different values of α and β, where d_N is a suitably chosen normalization factor and [ ] denotes the greatest integer. They show that $Z_N(t)$ converges (in $D[0,1]$) to a new type of non-Gaussian self-similar process which can be represented as a Wiener-Itô integral on $\mathbf{R}^2$. The representation is given explicitely and compared with that of other known processes.

The case of moving averages

The results obtained in Terrin and Taqqu (1989b) are extended in Terrin and Taqqu (1989c) to observations

$$X_j = \sum_{s=-\infty}^{\infty} b(j-s)\xi_s,\ j = 0, 1, \ldots$$

that are moving averages of i.i.d. random variables ξ_s's with finite variance. The coefficients $b(j)$'s are such that the X_j's exhibit long-range dependence. Moreover, the product X_jX_k in the quadratic form (1) is replaced by a more general Appell polynomial $P_{m,n}(X_j, X_k)$ of the observation, that is, a polynomial of degree at most $m+n$, which is adapted to the distribution of the X_j's. For example, $P_{m,n}$ is a Hermite polynomial if the X_j's are Gaussian. These polynomials characterize outputs of non-linear filters.

The asymptotic behavior as $N \to \infty$ of

$$\frac{1}{d_N} \sum_{j=1}^{N} \sum_{k=1}^{N} a_{j-k}\ P_{m,n}(X_j, X_k)$$

is then obtained. When the parameters are in a suitable range, the limit turns out to be essentially similar to the one which arises when $P_{m,n}(X_j, X_k) = X_jX_k - EX_jX_k$ with Gaussian X_j's.

Terrin and Taqqu (1989e) describe of the state of our present knowledge on the asymptotic behavior of quadratic forms.

Convergence to a Gaussian limit as the normalization exponent tends to 1/2

Certain quadratic forms with long-range dependence, normalized by N^d with $d > 1/2$, have a non-Gaussian limit. Terrin and Taqqu (1989d) show that under further normalization, as $d \to 1/2$, the limit becomes Gaussian. This illustrates the importance of the exponent in the normalization.

A generalized Hölder inequality

Avram and Taqqu (1989) extend the well-known Hölder inequality to functions of linearly dependent arguments. They show that in many situations, one has

$$\left| \int \prod_{j=1}^{n} f_j(l_j(x_1, \ldots, x_m)) d\mu(x_1) \ldots d\mu(x_m) \right| \leq K \prod_{j=1}^{n} \|f_j\|_{p_j}$$

where l_j are real-valued linear functions, $1 \leq p_j \leq \infty$ and $\| \ \|_p$ is the $L_p(\mu)$ norm. This inequality is useful in studying the behavior of Toeplitz matrices and has applications to maximum likelihood estimation.

Joint moments of stable random variables

Let $X_1, X_2, \ldots, X_n$ be jointly α-stable random variables, $0 < \alpha < 2$, and let $p_1, p_2, \ldots, p_n$ be non-negative numbers. When is $E|X_1|^{p_1} \ldots |X_n|^{p_n} < \infty$?

If the X_j are independent then $E|X_1|^{p_1} \ldots |X_n|^{p_n} = E|X_1|^{p_1} \ldots E|X_n|^{p_n} < \infty$ if and only if $p_j < \alpha$, $j = 1, \ldots, n$. If the X_j are all equal (say $X_j = X_1$, $j = 1, \ldots, n$), then $E|X_1|^{p_1} \ldots |X_n|^{p_n} = E|X_1|^{p_1 + \ldots + p_n} < \infty$ if and only if $p_1 + \ldots + p_n < \alpha$.

Samorodnitsky and Taqqu (1989c) give a necessary and sufficient condition for $E|X_1|^{p_1} \ldots |X_n|^{p_n}$ to be finite in the general case.

Conditional moments and linear regression for stable random variables

Jointly α-stable random variables with index $0 < \alpha < 2$ have only finite moments of order less than α. Samorodnitsky and Taqqu (1989b) show that their conditional moments can be higher than α. They provide conditions for this to happen and use the existence of the conditional moments to study the regression $E(X_2|X_1 = x)$. They show that if (X_1, X_2) is a symmetric α-stable random vector, then under appropriate conditions, the regression is well-defined even when $\alpha \leq 1$ and is linear in x.

More specifically, let Γ be the spectral measure of (X_1, X_2) and let S_2 denote the unit circle. If

$$\int_{S_2} \frac{\Gamma(ds)}{|s_1|^{\nu}} < \infty$$

for some $\nu > 0$, then the conditional moment $E(|X_2|^p|X_1 = x)$ is finite for p up to $\alpha + \nu$, with the restriction that $p < \alpha + 1$ if $0 < \alpha < 1$ and $p \leq 2$ if $1 \leq \alpha < 2$.

This provides conditions for the existence of the regression $E(X_2|X_1 = x)$. The regression is linear when it exists and is given by

$$E(X_2|X_1 = x) = \frac{[X_2, X_1]_\alpha}{\|X_1\|_\alpha^\alpha} x \quad a.s. \tag{2}$$

where $[X_2, X_1]_\alpha = \int_{S^2} s_2 s_1^{<\alpha-1>} \Gamma(ds)$ is the covariation of X_2 and X_1 and $\|X_1\|_\alpha^\alpha = [X_1, X_1]_\alpha$. Here $s_1^{<\alpha-1>} = |s_1|^{\alpha-1}$ if $s_1 > 0$ and $s_1^{<\alpha-1>} = -|s_1|^{\alpha-1}$ if $s_1 < 0$. The relation (2) is the natural extension of the well-known regression formula in the Gaussian case. It reduces to that formula if α is set equal to 2.

The results are applied to different classes of symmetric α-stable processes, e.g. moving averages, sub-Gaussian, harmonizable and sub-stable stochastic processes.

The various linear fractional Lévy motions

Linear Fractional Lévy motions are α-stable self-similar processes with stationary increments and a "moving average" representation. They can exhibit at the same time long-range dependence and high variability. Their representation

$$X(t) = \int_{-\infty}^{+\infty} \{a[(t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}] + b[(t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha}]\} M(dx)$$

involves two real parameters a and b. (The random measure M is α-stable.) When $\alpha = 2$, the processes are identical to the Gaussian Fractional Brownian motion for all values of a and b.

Samorodnitsky and Taqqu (1989a) show that when $0 < \alpha < 2$, then different values of the parameters a and b yield different processes, in marked contrast with the Gaussian case $\alpha = 2$. These results hold, even if the processes are non-symmetric.

The asymptotic dependence of the various linear fractional Lévy motions

Let $\{X(t),\ -\infty < t < \infty\}$ be the linear fractional Lévy motion, that is, an α-stable H-self-similar process with stationary increments, parametrized by $0 < \alpha < 2$ and $0 < H <$

1, $H \neq 1/\alpha$, and possessing a moving-average type representation. Let $Y(t) = X(t+1) - X(t)$ be the increment process. The asymptotic dependence structure of the stationary process $Y(\cdot)$ can be measured for real θ_1 and θ_2 by

$$r(t) := r_Y(\theta_1, \theta_2; t)$$

$$= E\exp\{i(\theta_1 Y(t) + \theta_2 Y(0))\} - E\exp\{i\theta_1 Y(t)\} E\exp\{i\theta_2 Y(0)\},$$

as $|t| \to \infty$. Astrauskas, Levy and Taqqu (1989) show that $r(t)$ decays like a power, i.e.

$$r(t) \sim A|t|^{\kappa}$$

$|t| \to \infty$, where $A = A(\theta_1, \theta_2, X, \alpha, H)$ and $\kappa = \kappa(\alpha, H) < 0$. The constants A and κ are obtained explicitely. The dependence of the exponent κ on H can be different in different regions. For example, if $1 < \alpha < 2$, then

$$r(t) \sim A|t|^{\alpha H - \alpha} \text{ if } 1 - 1/\alpha(\alpha - 1) < H < 1,\ H \neq 1/\alpha$$

but

$$r(t) \sim A_2|t|^{H-(1/\alpha)-1} \text{ if } 0 < H < 1 - 1/\alpha(\alpha - 1).$$

The behavior of $\lim_{|t|\to\infty} r(t)$ is also studied for the stationary increments of Log-fractional Lévy motion, an α-stable $1/\alpha$-self-similar process defined for $1 < \alpha < 2$.

Association

Random variables $X_1, \ldots, X_n$ are said to be *associated* if for any functions $f, g : \mathbf{R}^n \to \mathbf{R}$, nondecreasing in each argument, one has

$$\mathrm{Cov}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0$$

whenever the covariance exists. Lee, Rachev and Samorodnitsky (1989) give a necessary and sufficient condition for α-stable random variables $X_1, \ldots, X_n$ with $0 < \alpha < 2$ to be associated. The condition can be expressed in terms of their spectral measure Γ as follows.

Let S_n denote the unit sphere of $\mathbf{R}^n$ and let

$$S_n^- = \{(s_1, \ldots, s_n) \in S_n : \text{ for some } i, j, \in \{1, \ldots, n\},\ s_i > 0 \text{ and } s_j < 0\}.$$

The α-stable random variables $X_1, \ldots, X_n$ are associated if and only if Γ gives no mass to S_n^-.

Multiple stable integrals

Much progress has been achieved in constructing multiple stable integrals

$$\mathbf{I}_n(\mathbf{f}) = \int_{\mathbf{R}^n} \mathbf{f}(x_1, \ldots, x_n) M(dx_1) \ldots M(dx_n) \tag{3}$$

and in deriving their properties. Such integrals represent the output of a n-order filter (characterized by $\mathbf{f}$) with stable noise input.

Consider a symmetric α-stable, $0 < \alpha < 2$, random measure M with finite control measure on the real line. Let $\mathbf{f}$ be a Banach-valued deterministic function defined on $\mathbf{R}^n$, symmetric in its arguments and vanishing on the diagonals. Samorodnitsky and Taqqu (1989a) provide a construction of the product measure

$$M^{(n)}(dx_1, \ldots, dx_n) = n! M(dx_1) \ldots M(dx_n)$$

and of the multiple stochastic integral $\mathbf{I}_n(\mathbf{f})$ using a multiple LePage-type series representation.

The assumption that $\mathbf{f}$ belongs to a Banach space allows for many types of vector valued functions $\mathbf{f}$. If the Banach space is of the Rademacher-type $p > \alpha$ and if $\|\mathbf{f}\|$ satisfies an integrability condition, then the tail probability of $\|\mathbf{I}_n(\mathbf{f})\|$ is given by

$$P(\|\mathbf{I}_n(\mathbf{f})\| > \lambda) \sim C_{\alpha,n}(\mathbf{f}) \lambda^{-\alpha} (\ln \lambda)^{n-1}$$

as $\lambda \to \infty$. This last result is established in Samorodnitsky and Taqqu (1989b) and is very useful because little is known about the distribution properties of the multiple integral. In the case $n = 2$ and $S =$ the real line, the integrability condition reduces to the known necessary and sufficient condition for the existence of the integral.

Sample path properties of stochastic processes represented as multiple stable integrals

The paper Rosinski, Samorodnitsky and Taqqu (1989) studies the sample path properties of stochastic processes represented by multiple symmetric α-stable integrals as in (3). It relates the "smoothness" of the sample paths to the "smoothness" of the (non-random) integrand. It also contains many applications, including zero-one laws and the tail behavior of the distribution of suprema.

Approximating both the information structure and the paths of stochastic processes

Willinger and Taqqu (1988) describe a method for approximating continuous-time stochastic processes with discrete time and discrete state space models which possess the same "structural" properties as the original continuous-time model. With the help of the approximation, one can explain and study the structural properties of the continuous-time models (e.g. "completeness" or "martingale representation property") by using the corresponding simpler and well-understood properties of models that involve only finitely many time points and that can take only a finite number of different values.

One essential feature of the approximation scheme is that convergence is understood in a strong sense, not only for the approximating discrete processes (where pathwise convergence is required) but also for their corresponding information-structures (filtrations) for which a natural convergence concept is introduced. This strong double-edged form of convergence is responsible for the desirable "structure-preserving" property of the approximation scheme. The approach is illustrated with various examples of well-known continuous securities market models such as the Black-Scholes model.

PUBLICATIONS

[1] Astrauskas, A., Levy, J.B. and Taqqu, M.S. (1989). The asymptotic dependence structure of the linear fractional Lévy motion. To appear in the *Lithuanian Math. Journal.*

[2] Avram, F. and Taqqu, M.S. (1989). Hölder's inequality for functions of linearly dependent arguments. *SIAM Journal of Mathematical Analysis* **20**, No.6, 1484-1489.

[3] Dehling, H. and Taqqu, M.S. (1988a). Continuous functions whose level sets are orthogonal to all polynomials of a given degree. To appear in *Acta Mathematica Hungarica.*

[4] Dehling, H. and Taqqu, M.S. (1988b). Bivariate symmetric statistics of long-range dependent observations. To appear in the *Journal of Statistical Planning and Inference.*

[5] Rosinski, J., Samorodnitsky, G. and Taqqu, M.S. (1989). Sample path properties of stochastic processes represented as multiple stable integrals. Preprint.

[6] Samorodnitsky, G. and Taqqu, M.S. (1988a). Construction of multiple stable measures and integrals using LePage Representation. Preprint.

[7] Samorodnitsky, G. and Taqqu, M.S. (1988b). Multiple stable integrals of Banach-valued functions. To appear in the *Journal of Theoretical Probability*, Vol. **3**.

[8] Samorodnitsky, G. and Taqqu, M.S. (1989a). The various linear fractional Lévy motions. In: *Probability, Statistics and Mathematics: Papers in Honor of Samuel Karlin*, T.W. Anderson, K.B. Athreya and D.L. Iglehart, editors, 261-270. Boston: Academic Press.

[9] Samorodnitsky, G. and Taqqu, M.S. (1989b). Conditional moments and linear regression for stable random variables. Preprint.

[10] Samorodnitsky, G. and Taqqu, M.S. (1989c). Existence of joint moments of stable random variables. To appear in *Statistics and Probability Letters*, **8**, No.5.

[11] Terrin, N. and Taqqu, M.S. (1989a). Power counting theorem on $\mathbf{R}^n$. Preprint.

[12] Terrin, N. and Taqqu, M.S. (1989b). A noncentral limit theorem for quadratic forms of Gaussian stationary sequences. To appear in the *Journal of Theoretical Probability*.

[13] Terrin, N. and Taqqu, M.S. (1989c). Convergence in distribution of sums of bivariate Appell polynomials with long-range dependence. Preprint.

[14] Terrin, N. and Taqqu, M.S. (1989d). Convergence to a Gaussian limit as the normalization exponent tends to 1/2. Preprint.

[15] Willinger, W. and Taqqu, M.S. (1988). An approximation theory for continuous stochastic securities market models. Preprint.